Skew Braces, the Yang-Baxter Equation, Rings, and Hopf-Galois Structures

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Overview

Aim of this talk is to give an overview of

skew braces in group theory

and their application in

1. Mathematical physics
2. Ring theory
3. Number theory.

Skew Braces

Definition (Skew Brace)

A (left) **skew brace** is a triple $(B, \oplus, \odot)$ which consists of a set B together with two operations $\oplus$ and $\odot$ such that $(B, \oplus)$ and $(B, \odot)$ are groups such that for all $a, b, c \in B$:

$$a \odot (b \oplus c) = (a \odot b) \ominus a \oplus (a \odot c),$$

where $\ominus a$ is the inverse of a with respect to the operation $\oplus$.

Interesting for ring theorists:

$$a \odot 0 = a \odot (0 \oplus 0) = (a \odot 0) \ominus a \oplus (a \odot 0)$$

$$\Longrightarrow 0 = 1.$$

Skew Braces

Example

Any group $(B, \oplus)$ with

$$a \odot b = a \oplus b \quad \text{(similarly with } a \odot b = b \oplus a\text{)}$$

is a skew brace. This is the **trivial** skew brace structure.

A skew brace $(B, \oplus, \odot)$ is called a **brace** if $(B, \oplus)$ is abelian.

Braces were introduced by Rump [Rum07a] to **generalise radical rings**.

They provide *non-degenerate, involutive* **set-theoretic solutions of the Yang-Baxter equation**.

Skew Braces

Skew braces were introduced by Guarnieri and Vendramin [GV17] and **generalise braces**.

They provide *non-degenerate* **set-theoretic solutions of the Yang-Baxter equation**.

Their connection to **ring theory** and **Hopf-Galois structures** was studied by Byott, Konovalov, Smoktunowicz, and Vendramin [SV18, KSV18].

Skew Braces

Notation

we call a skew brace $(B, \oplus, \odot)$ such that $(B, \oplus) \cong N$ and $(B, \odot) \cong G$ a G-skew brace of **type** N.

- $(B, \oplus)$ is called the **additive group**.
- $(B, \odot)$ is called the **multiplicative group**.

Skew Braces: Some Results

- Rump (2007) classified **cyclic braces** [Rum07b].
- Bachiller (2015) classified **braces of order** p^3 [Bac15].
- Bachiller, Cedo, Jespers, Okninski (2017) **matched products of braces**.
- Guarnieri, Vendramin (2017) made conjectures using **computer assisted results** about certain skew braces [GV17].
- NZ (2018) **skew braces of order** p^3 [NZ18b, NZ18a].
- Skew braces with **non-trivial *annihilator*** [CCS18].
- Dietzel studied **braces of order** p^2q [Die18].

Skew Braces and Group Theory

Fix a skew brace of type N given by $(B, \oplus, \odot)$.

- $(B, \odot)$ **acts** on $(B, \oplus)$ by

$$(a, b) \longmapsto a \odot b.$$

- This gives a **regular embedding**

$$\begin{aligned} m : (B, \odot) &\longrightarrow \mathrm{Hol}\,(B, \oplus) \\ a &\longmapsto (m_a : \ b \longmapsto a \odot b)\,. \end{aligned}$$

- **Isomorphic** skew braces with additive group $(B, \oplus)$ give rise to **conjugate** regular subgroups.

Skew Braces and Regular Subgroups of Holomorph Correspondence

Bachiller, Byott, Vendramin:

$$\left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of skew braces of}\\ \text{type } N\text{, i.e., with}\\ (B,\oplus)\cong N\end{array}\right\} \overset{\text{bij}}{\leftrightsquigarrow} \left\{\begin{array}{c}\text{classes of regular subgroup of}\\ \mathrm{Hol}(N) \text{ under } H_1\sim H_2 \text{ if}\\ H_2=\alpha H_1\alpha^{-1} \text{ for some}\\ \alpha\in \mathrm{Aut}(N)\end{array}\right\}$$

Classifying Skew Braces

To find the non-isomorphic G-skew braces of type N classify elements of the set

$$\mathcal{S}(G,N)=\{H\subseteq \mathrm{Hol}\,(N)\mid H \text{ is regular, } H\cong G\},$$

and extract a maximal subset whose elements are not conjugate by any element of $\mathrm{Aut}\,(N)$.

Skew Braces of C_{p^n} type

Example

Let $p > 2$, $n > 1$, and $C_{p^n} = \left\langle \sigma \mid \sigma^{p^n} = 1 \right\rangle$. Then

$$\mathrm{Hol}\left(C_{p^n}\right) = \langle \sigma \rangle \rtimes \langle \beta, \gamma \rangle$$

with $\beta\left(\sigma\right) = \sigma^{p+1}$. Then the *trivial* (skew) brace is $\langle \sigma \rangle$, and the *nontrivial* (skew) braces are given by

$$\left\langle \sigma \beta^{p^m} \right\rangle \cong C_{p^n} \text{ for } m = 0, ..., n-2.$$

We also have

$$\mathrm{Aut}_{\mathcal{B}r}\left(\left\langle \sigma \beta^{p^m} \right\rangle\right) = \left\langle \beta^{p^{n-m-2}} \right\rangle \text{ for } m = 0, ..., n-2.$$

Skew Braces of Order p^3 for $p > 3$

The number of G-skew braces of type N, $\widetilde{e}(G, N)$, is given by

$\widetilde{e}(G,N)$	C_{p^3}	$C_{p^2} \times C_p$	C_p^3	$C_p^2 \rtimes C_p$	$C_{p^2} \rtimes C_p$
C_{p^3}	3	-	-	-	-
$C_{p^2} \times C_p$	-	9	-	-	$4p+1$
C_p^3	-	-	5	$2p+1$	-
$C_p^2 \rtimes C_p$	-	-	$2p+1$	$2p^2-p+3$	-
$C_{p^2} \rtimes C_p$	-	$4p+1$	-	-	$4p^2-3p-1$

Remark

Note

$$\widetilde{e}(G, N) = \widetilde{e}(N, G).$$

Skew braces provide solutions of the Yang-Baxter equation

The Yang-Baxter Equation

The **Yang-Baxter equation** appeared in works of Yang and Baxter in **statistical mechanics** and **mathematical physics**.

Nowadays the Yang-Baxter equation is studied in **quantum group theory** and has applications in **integrable systems**, **knot theory**, **tensor categories**, and other areas.

For a vector space V, the YBE is a matrix equation for elements of $\mathrm{GL}(V \otimes V)$.

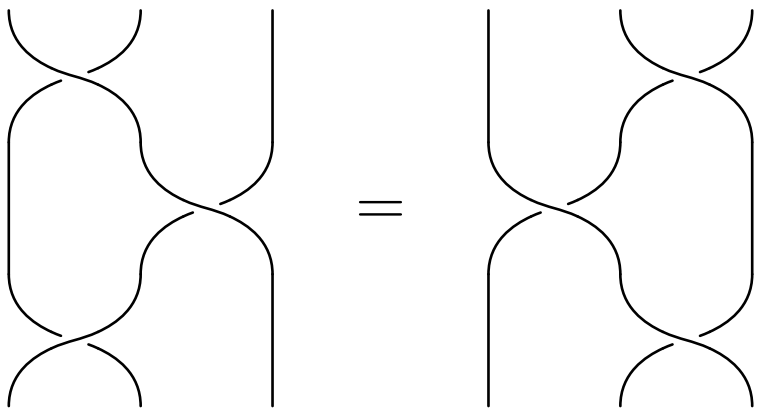
The Yang-Baxter Equation

An element $R \in \mathrm{GL}(V \otimes V)$ is said to satisfy **the Yang-Baxter equation** if

$$(R \otimes I)(I \otimes R)(R \otimes I) = (I \otimes R)(R \otimes I)(I \otimes R)$$

holds.

This equation can be represented by the picture



The Set-Theoretic Yang-Baxter Equation

In 1992 Drinfeld suggested studying the **simplest class of solutions** arising from the **set-theoretic** version of this equation.

Definition (Set-theoretic solution of the YBE)

Let X be a nonempty set and

$$\begin{aligned} r : X \times X &\longrightarrow X \times X \\ (x, y) &\longmapsto (f_x(y), g_y(x)) \end{aligned}$$

a bijection. Then (X, r) is a **set-theoretic solution** of YBE if

$$(r \times \mathrm{id})(\mathrm{id} \times r)(r \times \mathrm{id}) = (\mathrm{id} \times r)(r \times \mathrm{id})(\mathrm{id} \times r)$$

holds. (X, r) is **non-degenerate** if $f_x, g_x \in \mathrm{Perm}(X)$ for all $x \in X$ and **involutive** if $r^2 = \mathrm{id}$.

The Set-Theoretic YBE

Examples

Let X be a nonempty set.

1. (Trivial solution) The map $r(x, y) = (y, x)$.
2. (Permutations solutions) Let $f, g : X \longrightarrow X$ be bijections. Then

$$r(x, y) = (f(y), g(x))$$

gives a non-degenerate solution, which is involutive if and only if $f = g^{-1}$.

3. If X is a group, then

$$r(x, y) = (xyx^{-1}, x)$$

gives a solution.

Structure Group of (X, r)

Definition (Structure Group)

Let (X, r) be a non-degenerate solution of the YBE. The **structure group** of (X, r) is defined by

$$G(X, r) = \langle X \mid xy = f_x(y)g_y(x) \text{ for all } x, y \in X \rangle.$$

We have a natural map $\iota_G : X \longrightarrow G(X, r)$, which is not in general injective.

Example

For (X, r) the trivial solution, i.e., $r(x, y) = (y, x)$

$$G(X, r) = \mathbb{Z}^X.$$

It is known that $G(X, r)$ is **abelian-by-finite**.

Skew Braces and the YBE

Theorem (L. Guarnieri and L. Vendramin)

Let $(B, \oplus, \odot)$ be a skew brace. Then the map

$$r_B : B \times B \longrightarrow B \times B$$
$$(a, b) \longmapsto \left(\ominus a \oplus (a \odot b), (\ominus a \oplus (a \odot b))^{-1} \odot a \odot b\right)$$

is a non-degenerate set-theoretic solution of the YBE, which is involutive if $(B, \oplus, \odot)$ is a brace.

Furthermore, for every non-degenerate solution of YBE (X, r), the structure group $G(X, r)$ can be made into a skew braces in a unique way.

Skew braces generalise radical rings

Rings

Definition (Radical Ring)

A **radical ring** is a ring $(R, +, \cdot)$ which coincides with its own Jacobson radical, or equivalently if it becomes a group under the Jacobson circle operation

$$a \circ b = a + ab + b.$$

Definition (Two-sided Skew Brace)

A skew brace $(B, \oplus, \odot)$ is called **two-sided** if

$$(a \oplus b) \odot c = (a \odot c) \ominus c \oplus (b \odot c).$$

Two-sided Braces and Radical Rings

- Given a two-sided brace $(B, \oplus, \odot)$, define

$$a \otimes b = \ominus a \oplus (a \odot b) \ominus b.$$

  Then $(B, \oplus, \otimes)$ is a radical ring.
- Conversely, given a radical ring $(R, +, \cdot)$. Then $(R, +, \circ)$ is a two-sided brace.

Rump:

$$\{\text{two-sided braces}\} \overset{\text{bij}}{\leftrightsquigarrow} \{\text{ radical rings }\}$$

Skew and Other Ring Theoretic Structures

- For a skew brace $(B, \oplus, \odot)$ one can define

$$a \otimes b = \ominus a \oplus (a \odot b) \ominus b,$$

  and study the '*ring-like*' object $(B, \oplus, \otimes)$, for example see [KSV18].
- Skew braces are also connected to **near-rings** and nil rings.

Skew braces parametrise Hopf-Galois structures

Hopf-Galois Structures: Motivations

For simplicity we assume L/K is a **Galois extension** of fields with Galois group G.

Normal Basis Theorem

L is a free $K[G]$-module of rank one.

- Assume L/K is an extension of global or local fields (e.g., extensions of $\mathbb{Q}$ or $\mathbb{Q}_p$).
- Denote by $\mathcal{O}_L$ and $\mathcal{O}_K$ the rings of integers of L and K, respectively.
- Then $\mathcal{O}_L$ is also a module over $\mathcal{O}_K[G]$.

- Can $\mathcal{O}_L$ be free over $\mathcal{O}_K[G]$?

... No in general.

Hopf-Galois Structures

Hopf-Galois structures are K-Hopf algebras which act on L.

Definition (Hopf-Galois structure)

A **Hopf-Galois structure** on L/K consists of a finite dimensional cocommutative *K-Hopf algebra H* together with an action on L which makes L into an *H-Galois extension*.

The **group algebra** $K[G]$ endows L/K with the classical Hopf-Galois structure.

Hopf-Galois Structures: Applications

- Suppose H endows L/K with a Hopf-Galois structure.
- Define the associated order of $\mathcal{O}_L$ in H by

$$\mathfrak{A}_H = \{\alpha \in H \mid \alpha(\mathcal{O}_L) \subseteq \mathcal{O}_L\}.$$

- Can $\mathcal{O}_L$ be free over $\mathfrak{A}_H$?

  ... Sometimes, and depends on H.

Need a classification of Hopf-Galois structures.

Hopf-Galois Structures: A Theorem of Greither and Pareigis

Theorem (Greither and Pareigis [GP87])

Hopf-Galois structures on L/K correspond bijectively to regular subgroups of $\mathrm{Perm}(G)$ *which are normalised by the image of G, as left translations, inside* $\mathrm{Perm}(G)$.

Every K-Hopf algebra which endows L/K with a Hopf-Galois structure is of the form $L[N]^G$ for some regular subgroup $N \subseteq \mathrm{Perm}(G)$ normalised by the left translations.

Notation: The *isomorphism type* of N is known as the **type** of the Hopf-Galois structure.

Hopf-Galois Structures: Some Results

- Byott (1996) showed if $|G| = n$, then L/K a **unique Hopf-Galois structure** iff $\gcd(n, \phi(n)) = 1$ [Byo96].
- Kohl (1998) classified Hopf-Galois structures for $G = C_{p^n}$ for a prime $p > 2$ [Koh98]: there are p^{n-1}, all are of cyclic type. Byott (2007) studies $G = C_{2^n}$ case [Byo07].
- Byott (1996, 2004) studied the problem for $|G| = p^2, pq$, also when G is a nonabelian simple group [Byo96, Byo04].
- Carnahan and Childs (1999, 2005) studied Hopf-Galois structures for $G = C_p^n$ and $G = S_n$ [CC99].
- Alabadi and Byott (2017) studied the problem for $|G|$ is **squarefree** [AB18].
- NZ (2018) Hopf-Galois structures for $|G| = p^3$ [NZ18b].

Hopf-Galois Structures of Order p^3 for $p > 3$

The number of Hopf-Galois structures on L/K of type N, $e(G, N)$, is given by

$e(G,N)$	C_{p^3}	$C_{p^2} \times C_p$	C_p^3	$C_p^2 \rtimes C_p$	$C_{p^2} \rtimes C_p$
C_{p^3}	p^2	-	-	-	-
$C_{p^2} \times C_p$	-	$(2p-1)p^2$	-	-	$(2p-1)(p-1)p^2$
C_p^3	-	-	$(p^4+p^3-1)p^2$	$(p^3-1)(p^2+p-1)p^2$	-
$C_p^2 \rtimes C_p$	-	-	$(p^2+p-1)p^2$	$(2p^3-3p+1)p^2$	-
$C_{p^2} \rtimes C_p$	-	$(2p-1)p^2$	-	-	$(2p-1)(p-1)p^2$

Remark

Note $p^2 \mid e(G, N)$ and

$$e(G,N) = \frac{|\mathrm{Aut}(G)|}{|\mathrm{Aut}(N)|} e(N,G).$$

Skew Braces and Hopf-Galois Structures

Fix a G-skew brace given by $(B, \oplus, \odot)$.

- $(B, \oplus)$ **acts** on $(B, \odot)$ by

$$(a, b) \longmapsto a \oplus b.$$

- This gives a **regular embedding**

$$\begin{aligned} d : (B, \oplus) &\longrightarrow \mathrm{Perm}\,(B, \odot) \\ a &\longmapsto (d_a : \ b \longmapsto a \oplus b)\,, \end{aligned}$$

  whose image is **normalised** by the left translations.

- **Isomorphic** skew braces with multiplication group $(B, \odot)$ give rise to **conjugate** regular subgroups.

Skew Braces and Hopf-Galois Structures Correspondence

$$\left\{\begin{array}{c}\text{isomorphism classes}\\ \text{of } G\text{-skew braces,}\\ \text{i.e., with } (B,\odot)\cong G\end{array}\right\} \leftrightsquigarrow \left\{\begin{array}{c}\text{classes of Hopf-Galois structures}\\ \text{on } L/K \text{ under } L[N_1]^G \sim L[N_2]^G\\ \text{if } N_2 = \alpha N_1 \alpha^{-1} \text{ for some}\\ \alpha \in \mathrm{Aut}(G)\end{array}\right\}$$

Thank you for your attention!

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